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Mathematical analysis of the smallest chemical reaction system with Hopf bifurcation

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Recently we presented an up to now unstudied three-dimensional dynamical system which is, according to our given definition, the smallest chemical reaction system with Hopf bifurcation. We here study the Hopf bifurcation in detail and prove that near the bifurcation point a stable limit cycle arises. In the analysis we use the methods of local bifurcation theory, especially the center manifold and the normal form theorem. In a similar way we analyse the also occurring transcritical bifurcation. Besides studying local stability, we give the proofs for global stability of the trivial steady state in the whole positive phase space and for the nontrivial steady state in a closed domain containing the steady state point.

1. Introduction

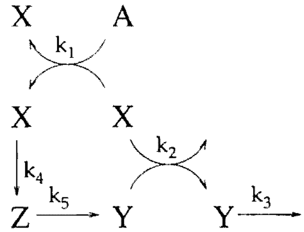
In a previous paper [1] we presented the smallest chemical reaction system with Hopf bifurcation. Its mechanism is depicted in scheme 1. In the definition for what we do mean by the "smallest" system we stressed the fact that the system is the mathematically most simple one with this kind of dynamic behavior. So it is the only three-component chemical reaction system with Hopf bifurcation that contains besides linear terms only one quadratic nonlinearity in its differential equations:

$$\dot{x} = kx - k_2xy\,,$$

$$\dot{y} = -k_3y + k_5z\,,$$

$$\dot{z} = k_4x - k_5z \qquad (1)$$

with $k = k_1A - k_4$ and $k_i > 0$ where k_i and A denote rate constants and the fixed concentration of the outer reactant of the autocatalytic reaction, respectively. Due to its simplicity the system is well suited for further mathematical treatment.



Scheme 1. Reaction scheme of the smallest chemical reaction system with Hopf bifurcation.

All (quantitative) information about the system would be gained if one could integrate it in an analytical manner. Because of the nonlinearity in the differential equations there is no general algorithm to do so. One can write the nonlinear system (1) of three differential equations of first order in the form of one nonlinear differential equation of third order:

$$\dddot{x} = \ddot{x}(-k_3 - k_5 + 3\dot{x}/x) + \dot{x}(-k_3k_5 + \dot{x}/x(k_3 + k_5) - 2(\dot{x}/x)^2) + k_5x(kk_3 - k_2k_4x)\,, \tag{2}$$

but no analytical solution for this system is known, neither in the form of (1) nor (2) (cf. e.g. [2]). With numerical integration we can obtain a picture of the system behavior which is in the most cases correct. (A well-known example for a wrong numerical result is the integration of the oscillator with feedback inhibition given by Goodwin (cf. [3]).)

In [1] we have shown numerically that system (1) possesses a stable limit cycle. Of course it is important to give an analytical proof for this result. A detailed analysis of the Hopf bifurcation, using the methods of local bifurcation theory, especially the center manifold and the normal form theorem (cf. [4]), proves that a stable limit cycle arises. In a similar way we do also study the second occurring steady state bifurcation, the transcritical bifurcation.

Besides detecting and analysing the bifurcation points it is important to study the global stability in order to obtain a reliable right picture of the qualitative system behavior. For the parameter region where the trivial steady state ($\bar{x} = \bar{y} = \bar{z} = 0$) is stable we give a simple Lyapunov function to show that this state is also globally stable. Although numerical integrations indicate that also the nontrivial steady state is globally stable in the region of local stability we could not find a suitable Lyapunov function for proving this result for the whole positive phase space. However, we are able to give a Lyapunov function which proves the global stability in a certain domain containing the steady state point. Furthermore, we show that for proving the global stability in the whole positive phase space an appropriate Lyapunov function must be more complicated than a surface described by only first or second order terms.

2. Local analysis

2.1. LINEAR STABILITY ANALYSIS

The system (1) has two steady states:

1. $\bar{x}_1 = \bar{y}_1 = \bar{z}_1 = 0\,,$ (3)

2. $\bar{x}_2 = kk_3/(k_2k_4), \quad \bar{y}_2 = k/k_2, \quad \bar{z}_2 = kk_3/(k_2k_5)\,.$ (4)

It follows from the Hurwitz criterion that the first steady state is stable within the range $-k_4 < k < 0$ and the second one within the range $0 < k < k_3 + k_5$ (cf. [1]). The bifurcation diagram is given in fig. 1.

The system has two bifurcation points. At $k = 0$ there only exists the trivial steady state at which the Jacobian has two real negative eigenvalues and one which equals zero. In the bifurcation diagram it can be seen that this point seems to be a point of a transcritical bifurcation. At the second bifurcation point $k = k_3 + k_5$ the trivial steady state is locally asymptotically unstable whereas the Jacobian at the positive steady state has one real negative and two purely imaginary eigenvalues, so it seems to be a Hopf bifurcation point. At both bifurcation points the Jacobian of the system has at least one eigenvalue with a real part equal to zero so that one cannot deduce the local stability behavior from the linear analysis.

2.2. ANALYSIS OF THE SYSTEM AT THE BIFURCATION POINTS WITH THE METHODS OF THE LOCAL BIFURCATION THEORY

A general aim of bifurcation theory is to get a classification of as many as possible different types of bifurcations. The theory is fairly complete only for the so-called codimension one and two bifurcations which can be contained in a parameter space of dimension at least one and two, resp. (see [4]). Both the transcritical and the Hopf bifurcation are codimension one bifurcations, so the detailed analysis could be simplified by using the already known general theorems (cf. [5,6]). Never-

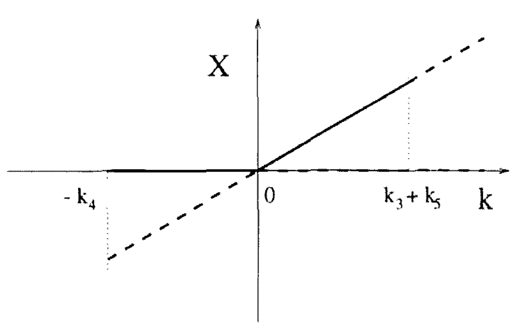


Fig. 1. Bifurcation diagram for system (1). - - - unstable, —— stable steady states.

theless we want to analyse both bifurcations in a unified manner using the methods of local bifurcation theory. At first we reduce the dimension of the problem at the bifurcation point using the center manifold theorem. In this way one can already extract all relevant information about the transcritical bifurcation. For simplifying the flux in the center manifold the normal form theorem is used in order to remove all nonnecessary terms up to a certain order while retaining the right qualitative behavior of the system at the bifurcation point. The result of this simplification is the so-called normal form of the flux which makes for the codimension two bifurcations and the Hopf bifurcation some symmetry properties of the bifurcation apparent [4]. The normal form of the flux in the center manifold at a Hopf bifurcation point contains for example no quadratic terms and is symmetric under the transformation $(x, y) \to (-y, x)$.

2.2.1. The transcritical bifurcation

In this section we analyse the system behavior at the point $k = 0$. At this bifurcation point the system has only the trivial steady state $\bar{x} = \bar{y} = \bar{z} = 0$. The eigenvalues of the Jacobian at this point are

$$\lambda_1 = 0; \quad \lambda_2 = -k_3; \quad \lambda_3 = -k_5\,.$$

Using the eigenvectors as the basis for a new coordinate system, equation system (1) can be transformed with the linear coordinate transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = T \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

which involves the transformation matrix

$$T = \begin{pmatrix} \frac{k_5}{k_4} & 0 & 0 \\ \frac{k_5}{k_3} & 1 & 1 \\ 1 & 0 & \frac{k_3-k_5}{k_5} \end{pmatrix} \tag{5}$$

into the system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ \frac{k_5^2 k}{k_3^2 - k_3 k_5} & -k_3 & 0 \\ \frac{k_5 k}{k_5 - k_3} & 0 & -k_5 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} - k_2 u \left(\frac{k_5}{k_3} u + v + w \right) \begin{pmatrix} 1 \\ \frac{k_5^2}{k_3^2 - k_3 k_5} \\ \frac{k_5}{k_5 - k_3} \end{pmatrix}, \tag{6}$$

where the linear part at $k = 0$ is in standard (diagonal) form. After adding $\dot{k} = 0$ as a fourth differential equation to system (6) it follows from the center manifold theorem (cf. [4]) that at (0, 0, 0, 0) exists a two-dimensional center manifold W^c tangent to the plane spanned by the u- and k-axis. Therefore, W^c can be approximated for the two variables v, w by the equations

$$v(u,k) = \alpha_v u^2 + \beta_v uk + \gamma_v k^2 + \mathcal{O}(3)\,, \tag{7}$$

$$w(u,k) = \alpha_w u^2 + \beta_w uk + \gamma_w k^2 + \mathcal{O}(3)\,,$$

where $\mathcal{O}(3)$ denotes terms of order u^3, u^2k, uk^2 and k^3. With

$$\dot{v}(u,k) = \frac{\partial v}{\partial u}\dot{u} + \frac{\partial v}{\partial k}\dot{k}\,, \tag{8}$$

$$\dot{w}(u,k) = \frac{\partial w}{\partial u}\dot{u} + \frac{\partial w}{\partial k}\dot{k}\,,$$

together with (6) and (7) after comparison of the coefficients for u^2, uk and k^2 follows the center manifold as an analytical expression for $v(u,k)$ and $w(u,k)$ up to second order. In this way one can approximate the center manifold up to any desired order. Inserting the so obtained expressions for v, w into the differential equation for u in system (6) one obtains approximately the flux in the center manifold which determines the dynamic behavior of the system in this case. Since the functions $v(u,k)$, $w(u,k)$ are at least expressions of second order which are in the equation for u in (6) multiplied by u they can give only terms of order three and higher. Therefore, the flux in the center manifold up to second order reads

$$\dot{u} = u\left(k - \frac{k_2 k_5}{k_3} u\right) + \mathcal{O}(3)\,, \tag{9}$$

$$\dot{k} = 0\,.$$

Eq. (9) expresses the normal form of a transcritical bifurcation, so it is proven that at the point $k = 0$ this bifurcation occurs. Sketching the flux in the center manifold according to (9) one obtains the bifurcation diagram of a transcritical bifurcation. From (9) it can also be seen that the bifurcation point itself is locally stable for positive values of u and unstable for negative values. It follows from the inverse of the transformation matrix T [eq. (5)] that $u = k_4/k_5 x$. Therefore, u is positive for all positive values of x. Since the trajectories are confined to the positive orthant, the trivial steady state is locally stable at the transcritical bifurcation point.

2.2.2. The Hopf bifurcation

For analysing system (1) at the second bifurcation point $k = k_3 + k_5$ we introduce for the sake of simplicity the new time scale $\tau = k_3 t$ to eliminate the parameter k_3. With $k/k_3 \to k$, $k_i/k_3 \to k_i$, $(i = 2, 4, 5)$ the equation system reads

$$\dot{x} = kx - k_2 xy\,,$$

$$\dot{y} = -y + k_5 z\,,$$

$$\dot{z} = k_4 x - k_5 z\,. \tag{10}$$

The nontrivial steady state (4) at which the Hopf bifurcation occurs now reads

$$\bar{x} = k/(k_2 k_4); \quad \bar{y} = k/k_2; \quad \bar{z} = k/(k_2 k_5)\,. \tag{11}$$

[cf. eq. (4)]. The Jacobian of (10) at (11) follows as

$$J = \begin{pmatrix} 0 & -k/k_4 & 0 \\ 0 & -1 & k_5 \\ k_4 & 0 & -k_5 \end{pmatrix}. \tag{12}$$

If the parameters fulfill the condition $k = 1 + k_5$ this Jacobian has one negative real and two purely imaginary eigenvalues:

$$\lambda_1 = -1 - k_5; \quad \lambda_{2/3} = \pm i\sqrt{k_5}\,, \tag{13}$$

so the first condition for a Hopf bifurcation in the sense of the theorem given, e.g., in [4,7] is fulfilled. Nevertheless, for applying the Hopf bifurcation theorem a second condition must be fulfilled. The conjugate complex eigenvalue which is imaginary at the bifurcation point, $\lambda_{2/3}(k)$, must, if the parameter k is varied and the other parameters are fixed, cross the imaginary axes in the simple way:

$$\frac{d}{dk}(\mathrm{Re}\,\lambda_{2/3}(k))|_{k=1+k_5} = d \neq 0\,, \tag{14}$$

where Re $\lambda_{2/3}(k)$ denotes the real part of λ which is a smooth function of k. From (12) follows the characteristic polynomial:

$$\lambda^3 + (1 + k_5)\lambda^2 + k_5\lambda + kk_5 = 0\,. \tag{15}$$

We now calculate d as defined in (14) without solving (15) explicitly. Inserting $\mu + i\omega$ for the complex eigenvalue into the characteristic polynomial (15) yields two independent equations for its real and imaginary parts. After implicit derivation with respect to k one obtains with $\mu = 0$ and $\omega = \sqrt{k_5}$ [cf. (13)] an inhomogeneous linear equation system which can easily be solved for $\omega' = d\omega/(dk)$ and the searched $d = \mu' = d\mu/(dk)$:

$$d = \frac{k_5}{2(1 + 3k_5 + k_5^2)} > 0\,. \tag{16}$$

Thus, also the second condition for a Hopf bifurcation is fulfilled and the Hopf bifurcation theorem holds. The frequency of the arising limit cycle follows from the imaginary eigenvalue of the Jacobian at the bifurcation point according to (13): $\omega|_{k=1+k_5} = \sqrt{k_5}$ which reads in the original parameters $\omega|_{k=k_3+k_5} = \sqrt{k_3 k_5}$.

We now analyse the Hopf bifurcation of system (10) in detail. At first we give an expression for the flux in the center manifold at the bifurcation point which is two-dimensional (W^c has the same dimension as the (generalized) eigenspace of the conjugate complex eigenvalue with zero real part.). Before using the center manifold theorem one has to translate the steady state (11) into the origin. In the new coordinates $x - \bar{x} \to x, y - \bar{y} \to y, z - \bar{z} \to z$ system (10) reads

$$\dot{x} = -k/k_4 y - k_2 xy\,,$$
$$\dot{y} = -y + k_5 z\,,$$
$$\dot{z} = k_4 x - k_5 z\,. \tag{17}$$

With

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = T \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

where

$$T = \begin{pmatrix} -\frac{1}{k_4} & \frac{k_5 - i\sqrt{k_5}}{k_4} & \frac{k_5 + i\sqrt{k_5}}{k_4} \\ -1 & \frac{k_5}{1-i\sqrt{k_5}} & \frac{k_5}{1+i\sqrt{k_5}} \\ 1 & 1 & 1 \end{pmatrix},$$

system (17) is transformed into the diagonalised system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} -1-k_5 & 0 & 0 \\ 0 & -i\sqrt{k_5} & 0 \\ 0 & 0 & i\sqrt{k_5} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + P \begin{pmatrix} -k_5/n_1 \\ 1/n_2 \\ 1/\bar{n}_2 \end{pmatrix} \tag{18}$$

with

$$P = -k_2(1+k_5)u^2 + auv + bv^2 + \bar{a}uw + \bar{b}w^2\,,$$
$$n_1 = 1 + 4k_5 + 4k_5^2 + k_5^3, n_2 = 2\sqrt{k_5}(-i - 2ik_5 + k_5^{3/2}) \tag{19}$$

with

$$a = \sqrt{k_5}k_2(-i + 2\sqrt{k_5} + k_5^{3/2}), \quad b = k_5^{3/2}k_2(i - ik_5 - 2\sqrt{k_5})\,, \tag{20}$$

where an overbar denotes complex conjugation. According to the center manifold theorem W^c is tangent to $E^c = \text{span}\{v, w\}$. Therefore one can approximate W^c with $u = h(v, w) = \alpha v^2 + \beta vw + \gamma w^2 + \mathcal{O}(3)$. In the same way as already described for the transcritical bifurcation one can find expressions for α, β, γ. After inserting $u = h(v, w)$ into the equations for v, w in (18) one obtains an approximated expression for the flux in the center manifold:

$$\begin{pmatrix} \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} -i\sqrt{k_5}v \\ i\sqrt{k_5}w \end{pmatrix} + (bv^2 + \bar{b}w^2 + auv + \bar{a}uw)\begin{pmatrix} 1/n_2 \\ 1/\bar{n}_2 \end{pmatrix} \tag{21}$$

with a, b, n_2 from (19), (20) and

$$u = cv^2 + \bar{c}w^2; \quad c = \frac{k_5^2 k_2 (4k_5 + i\sqrt{k_5}(-1 + 4k_5 + k_5^2))}{(1 + 6k_5 + k_5^2)(1 + 4k_5 + 4k_5^2 + k_5^3)} .$$

In the second step we now simplify the expression for the flux in the center manifold by removing all of the redundant nonlinear terms. The most simple expression is the normal form which still contains all information about the qualitative behavior of the system at the bifurcation point. With a further linear coordinate transformation, system (21) can be rewritten into a form which only contains real numbers giving the so-called standard form. With

$$\begin{pmatrix} v \\ w \end{pmatrix} = T \begin{pmatrix} \xi \\ \chi \end{pmatrix}; \quad T = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

it follows that

$$\begin{aligned} \dot{\xi} = & -\sqrt{k_5}\chi + 2k_5^2 k_2 (k_5(\chi^2 - \xi^2) + \sqrt{k_5}(1 - k_5)\xi\chi)/n_1 \\ & + 4k_5^4 k_2^2 (2k_5(2 + k_5)\xi^3 + \sqrt{k_5}(-4 + 7k_5 + 6k_5^2 + k_5^3)\xi^2\chi \\ & + (1 - 8k_5 - 3k_5^2)\xi\chi^2 + 2\sqrt{k_5}\chi^3)/n , \end{aligned} \tag{22}$$

$$\begin{aligned} \dot{\chi} = & \sqrt{k_5}\xi + 2k_5 k_2 (\sqrt{k_5}(1 + 2k_5)(\xi^2 - \chi^2) + (-1 - k_5 + 2k_5^2)\xi\chi)/n_1 \\ & + 4k_5^{5/2} k_2^2 ((-2k_5(2 + 5k_5 + 2k_5^2)\xi^3 + \sqrt{k_5}(4 + k_5 - 20k_5^2 - 13k_5^3 \\ & - 2k_5^4)\xi^2\chi + (-1 + 6k_5 + 19k_5^2 + 6k_5^3)\xi\chi^2 - 2\sqrt{k_5}(1 + 2k_5)\chi^3)/n \end{aligned}$$

with n_1 from (19), $n = (1 + 6k_5 + k_5^2)(1 + k_5)^2(1 + 3k_5 + k_5^2)^2$. Guckenheimer and Holmes [4] have explicitly shown that on the basis of the normal form theorem one finds a nonlinear coordinate transformation which transforms every system with the structure

$$\begin{aligned} \dot{\xi} &= -\omega\chi + o(|\xi|, |\chi|) , \\ \dot{\chi} &= \omega\xi + o(|\xi|, |\chi|) \end{aligned} \tag{23}$$

into the system

$$\begin{aligned} \dot{u} &= -\omega v + (au - bv)(u^2 + v^2) + \mathcal{O}(4) , \\ \dot{v} &= \omega u + (av + bu)(u^2 + v^2) + \mathcal{O}(4) , \end{aligned} \tag{24}$$

which is expressed in polar coordinates as

$$\begin{aligned} \dot{r} &= ar^3 , \\ \dot{\theta} &= \omega + br^2 . \end{aligned} \tag{25}$$

It can be seen that the sign of a determines the stability of the steady state at the Hopf bifurcation point. If one knows the general structure of the normal form then one can calculate the occurring coefficients in detail. Guckenheimer and Holmes carried out the procedure for calculating the coefficient a and gave the formula

$$a = 1/16(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} + 1/\omega(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}))\,, \tag{26}$$

where f_{xy} denotes $(\partial^2 f/\partial x \partial y)(0,0)$, etc. and f, g are the functions containing the nonlinear terms of eq. (23). In the same way one can find an expression for the coefficient b. It follows that

$$b = 1/16(g_{xxx} + g_{xyy} - f_{xxy} - f_{yyy} + 1/(3\omega)(5(f_{xx}g_{xy} + f_{xy}g_{yy} - f_{xx}f_{yy} - f_{yy}^2 - g_{xx}g_{yy} - g_{xx}^2) - 2(f_{xx}^2 + f_{xy}^2 + g_{xy}^2 + g_{yy}^2) + f_{yy}g_{xy} + f_{xy}g_{xx}))\,. \tag{27}$$

Applying eqs. (26), (27) to expression (22) which has the structure of (23) one obtains

$$a = \frac{-k_5^3 k_2^2}{(1 + 6k_5 + k_5^2)(1 + 3k_5 + k_5^2)} < 0\,, \tag{28}$$

$$b = \frac{-k_5^{3/2} k_2^2 (1 + k_5)(1 + 9k_5 + k_5^2)}{6(1 + 3k_5 + k_5^2)(1 + 6k_5 + k_5^2)} < 0\,. \tag{29}$$

It follows from $a < 0$ that the steady state at the bifurcation point is a locally (weak) stable point. Since $\omega = \sqrt{k_5} > 0$ and $b < 0$ it can be seen from (25) that there is a critical value $r = r^*$ where the direction of rotation changes.

In appendix A we show how the normal form can be calculated in a direct way, which of course gives the same results for the parameters a, b as (28), (29).

Up to now we have studied the system behavior exactly at the Hopf bifurcation point where one has to analyse the flux in a two-dimensional center manifold. However, the Hopf bifurcation theorem contains even more information, because it gives the flux in the three-dimensional center manifold containing the one parameter the bifurcation is depending on. Up to the third order the normal form in this case reads

$$\dot{x} = (d\mu + a(x^2 + y^2))x - (\omega + c\mu + b(x^2 + y^2))y\,, \tag{30}$$

$$\dot{y} = (\omega + c\mu + b(x^2 + y^2)x + (d\mu + a(x^2 + y^2))y\,,$$

where the parameters a, b follow from (26),(27) and d is calculated in the way of (14). At $\mu = 0$ the bifurcation occurs. The Hopf bifurcation theorem states that if the two conditions: (1) one pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts and (2) condition of equation (14) are both fulfilled then a three-dimensional center manifold exists. If, moreover, $a \neq 0$ then there is a surface of periodic solutions in W^c. If $a < 0$ then these periodic solutions are stable limit cycles (supercritical Hopf bifurcation), if $a > 0$ they are repelling (subcritical Hopf bifurcation). Since $a < 0$ [cf. eq. (28)] we can conclude that the Hopf bifurcation occurring in the original system (1) is for all allowed parameter values a supercritical one and the arising limit cycles are already locally stable. In polar coordinates, system (30) reads

$$\dot{r} = (d\mu + ar^2)r, \qquad (31)$$

$$\dot{\theta} = \omega + c\mu + br^2.$$

Because the $\dot{r}$ equation is separate from θ one can integrate it directly:

$$r = \sqrt{\frac{d\mu}{(a + \frac{d\mu}{r_0^2})e^{-2d\mu t} - a}}. \qquad (32)$$

Since the parameters a and d are known from (28), (16) we have an up to third order approximate expression for the radius of the limit cycle and the dynamics with which this limit cycle will be reached depending on the parameter μ.

3. Global stability analysis

3.1. THE TRIVIAL STEADY STATE

It follows from the local stability analysis that the trivial steady state is locally stable within the range $-k_4 \leqslant k \leqslant 0$. Just from system (1) it can be seen that for the only chemically interesting positive values of the variables it also should be globally stable: In the given parameter range $\dot{x}$ is always negative, i.e. starting with a positive value for x_0 the variable x will reach zero for $t \to \infty$. If x becomes very low, $\dot{z}$ becomes negative, so also z will reach zero and for the same reason also y. The mathematical proof for global stability, however, requires the finding of a suitable Lyapunov function. For the trivial steady state this is quite easy because a simple linear surface is already sufficient for proving the global stability: The function

$$V(x, y, z) = ax + by + cz \quad (a, b, c > 0) \qquad (33)$$

fulfills the presupposition for a Lyapunov function for proving the global stability of the trivial steady state for the whole positive orthant: $V(0, 0, 0) = 0$, $V(x, y, z) > 0$ for all $(x, y, z) \neq (\bar{x}, \bar{y}, \bar{z})$. With (1) it follows that

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} = (ak + ck_4)x - bk_3y + (b - c)k_5z - ak_2xy, \qquad (34)$$

so $\dot{V} < 0$ holds for all $(x, y, z) \neq (0, 0, 0)$ of the positive orthant if one chooses

$a = -(1+\epsilon_1)k_4/k + \epsilon_2$; $b = 1$; $c = 1+\epsilon_1$ ($\epsilon_1, \epsilon_2 > 0$). With $\epsilon_1 = 1$; $\epsilon_2 = -k_4/k$ one obtains a nice simple expression for the Lyapunov function (33):

$$V(x,y,z) = \frac{-3k_4}{k}x + y + 2z\,, \tag{35}$$

which proves the global stability of the trivial steady state for $k < 0$.

3.2. THE NONTRIVIAL STEADY STATE

The second steady state (4) of system (1) is locally stable within the parameter range $0 \leqslant k \leqslant k_3 + k_5$. Numerical integrations for different initial values in this parameter range suggest that it is also globally stable. Generally a Lyapunov function for a mathematical proof of global stability is at first searched among the class of surfaces of first or second order. It is now shown that one cannot prove the global stability for the whole positive orthant of the phase space with a quadratic surface. The most general form of it is

$$V(x,y,z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz\,. \tag{36}$$

A suitable Lyapunov function V has the properties $V(\bar{x},\bar{y},\bar{z}) = 0$, $V(x,y,z) > 0$ if $(x,y,z) \neq (\bar{x},\bar{y},\bar{z})$, which gives the following conditions for the quadratic surface (36):

$$g, h, i = 0\,, \tag{37}$$

$$a, b, c > 0\,, \tag{38}$$

$$4ab > d^2, 4ac > e^2, \quad 4bc > f^2\,, \tag{39}$$

$$4abc + def > cd^2 + be^2 + af^2\,. \tag{40}$$

Of course the center of the so described ellipsoid must correspond to the steady state. The function V now reads

$$\begin{aligned} V_s(x,y,z) = {} & a(x-\bar{x})^2 + b(y-\bar{y})^2 + c(z-\bar{z})^2 \\ & + d(x-\bar{x})(y-\bar{y}) + e(x-\bar{x})(z-\bar{z}) + f(y-\bar{y})(z-\bar{z}) \end{aligned} \tag{41}$$

and it follows with (1) that

$$\begin{aligned} \dot{V}_s = {} & x(\tilde{g}k + \tilde{i}k_4) - \tilde{h}k_3y + k_5z(\tilde{h} - \tilde{i}) + x^2(2ak + ek_4) - 2bk_3y^2 \\ & + k_5z^2(f - 2c) + xy(dk - \tilde{g}k_2 - dk_3 + fk_4) + xz(ek + 2ck_4 \\ & + k_5(d-e)) + yz(2bk_5 - f(k_3 + k_5)) - 2ak_2x^2y \\ & - dk_2xy^2 - ek_2xyz\,, \end{aligned} \tag{42}$$

where $\tilde{g} = -2a\bar{x} - d\bar{y} - e\bar{z}$, $\tilde{h} = -2b\bar{y} - d\bar{x} - f\bar{z}$, $\tilde{i} = -2c\bar{z} - e\bar{x} - f\bar{y}$. Global stability holds if the parameters a, b, c, d, e, f can be specified so that $\dot{V}_s < 0$ for all

$(x,y,z) \neq (\bar{x},\bar{y},\bar{z})$. A necessary condition for that is that the summands for x^2 and xyz in (42) both have a negative sign which for the interesting parameter range $k > 0$ can never be true. So it can be seen that a quadratic surface cannot serve as a Lyapunov function for proving the global stability of the nontrivial steady state for the whole positive phase space.

However, if one specifies a, b, c, d, e, f in (41) in the following way:

$$a = \frac{k_4((k_3+k_5)(k_5^2/k_3 + k_3^2/k_5 + k + k_5 + k_5/k) + k_3)}{2k_3k_5(k_3+k_5)(k_3+k_5-k)}, \tag{43}$$

$$b = \frac{k}{2k_4}\left(\frac{k(k_3/k_5 + k_5/k_3 + 1)}{k_5(k_3+k_5-k)} + \frac{2k_3+k_5}{k_3(k_3+k_5)}\right), \tag{44}$$

$$c = \frac{k(k_3/k_5 + k_5/k_3 + 1)}{2k_4(k_3+k_5-k)}, \tag{45}$$

$$d = -\frac{k(k_3/k_5 + k_5/k_3 + 1)}{k_5(k_3+k_5-k)} - \frac{1}{k_3+k_5}, \tag{46}$$

$$e = -\frac{1}{k_3+k_5}, \tag{47}$$

$$f = \frac{k(k_5^2 + kk_3)}{k_3k_4k_5(k_3+k_5-k)}, \tag{48}$$

one obtains a Lyapunov function which proves the global stability in certain finite surroundings of the steady state for the whole parameter region $0 < k < k_3 + k_5$. (Appendix B shows the way for finding these specifications.) Numerically one can find the extent of this region in calculating the scalar product of $grad(V)$ and $(\dot{x},\dot{y},\dot{z})$ for each point of the surface starting with small values of V and increasing V successively in small steps up to a value where at least one trajectory is directed outwards. Fig. 2 shows projections on the x,y-plane of the ellipsoids inside which the global stability is proven for six different values of the parameter k taken from the whole region $0 < k < k_3 + k_5$.

For the parameter region $k > k_3 + k_5$ it would be very interesting to find a confined set, i.e. an outwards impermeable closed surface, containing the limit cycle in its inside. On this basis it could be possible to prove the existence of the limit cycle for the parameter k in a finite distance from $k_3 + k_5$ (cf. [3]) and eventually for a certain domain the global stability of the limit cycle. The conditions on a general quadratic surface (36) for describing a closed surface V_{cs} are the same as for the Lyapunov function V_s (37)–(40) without conditions (37). For $\dot{V}_{cs}$ one therefore obtains again expression $\dot{V}_s$ (42) plus terms containing g, h, i. Now $\dot{V}_{cs} < 0$ must hold not for all but only for x, y, z belonging to the surface corresponding to the one specified, $V_{cs} = \text{const}$. Finding appropriate functions $a = f_1(k_i), \ldots, i = f_9(k_i)$ so

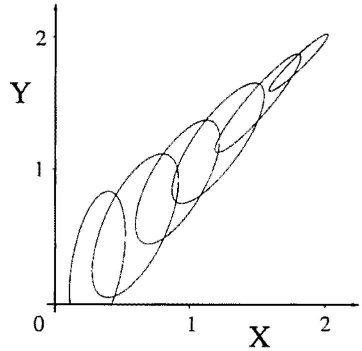


Fig. 2. Regions of proved global stability for six different parameter values: $k_2 = k_3 = k_4 = k_5 = 1$, $k = 0.3, 0.6, 0.9, 1.2, 1.5, 1.8$ (from left to right) as projections on the x, y-plane. The corresponding maximal values for V_s of (41) with the specifications (43)–(48) where still all trajectories are crossing the surface inwards are: 0.06, 0.12, 0.16, 0.16, 0.12, 0.07.

that this condition holds is, if at all possible, difficult. One therefore better use more refined methods such as, e.g., the construction of an auxiliary system of differential equations the solution of which describes the searched positive invariant set (see [8]).

4. Discussion

By analysing the Hopf bifurcation we have shown that the arising limit cycle is locally stable near the bifurcation point. This raises the question of local stability of the limit cycle at finite distances from the bifurcation point and for eventual global stability of the limit cycle. With numerical integration we found a globally stable limit cycle for the whole parameter region with two locally unstable steady states. Generally the question of local stability of a limit cycle is handled in the framework of the Floquet theory. Unfortunately, we are not able to use this theory because we do not have an analytical expression for the limit cycle to linearise the system around it. But even if we could do so for systems with more than two variables there are no general methods available for calculating the characteristic multipliers or exponents which are the relevant quantities for the local stability of the limit cycle.

One presupposition for proving global stability would be to find a closed surface containing the whole periodic orbit at which the vector field always points inward. Going on in this way Tyson [3] was able to prove the existence of a simple closed orbit for the Goodwin oscillator using Brouwer's fixed point theorem but not the stability itself. We have shown that it is too difficult to find such a surface among the class of quadratic surfaces and one should better use other methods for that purpose, such as the description of the closed invariant domain by the solution of an auxiliary system of differential equations (cf. [8]).

In [1] system (1) was defined as the smallest chemical reaction system with Hopf bifurcation in a mathematical sense. In our given definition for the smallest system, the feature "lowest number of quadratic terms in the first order differential equation system", which expresses mathematical simplicity, has a higher priority than "lowest number of reactions", which is obviously more important for simplicity in a physical sense. However, system (1) can be transformed into another, mathematically equivalent, form. All expressions which represent the system in the form of three first order differential equations, e.g. the diagonalised form at the trivial steady state [cf (6)], cannot be easier than (1) or (10), because the quadratic nonlinearity cannot be removed. It is interesting to note that system (1) expressed as one nonlinear third order differential equation [cf. (2)] looks quite complicated and it cannot be excluded that a chemical reaction system with Hopf bifurcation may exist which, if written in the form of one third order differential equation, is easier than (2) and perhaps even accessible for an analytical integration. However, usually one uses first order differential equations for expressing a chemical reaction system and employs also this form for a first mathematical analysis.

Unfortunately, there seems to be no way to integrate eq. (2) at least one time to decrease the order of this differential equation. If one could do so one could exclude a lot of the potentially possible dynamic phenoma in three-dimensional systems. Although numerically we did not find more complicated behavior than a simple closed orbit it remains an open question whether there are parameter values at which the system shows, e.g., folded orbits, quasiperiodicity or chaos.

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Appendix A

In this appendix we show how to calculate the normal form (24) with a nonlinear coordinate transformation directly.

The Jacobian of a differential equation system at a Hopf bifurcation point has two imaginary eigenvalues with a resonance of third order $\lambda_1 + \lambda_2 = 0$. If there are no second order resonant eigenvalues it follows from the theorem of Poincaré that with a suitable variable transformation all quadratic but not all cubic terms of this system can be eliminated. From the theorem of Poincaré–Dulaque it follows that it is possible to transform a two-dimensional system at a Hopf bifurcation point into the form given in eq. (24), which only contains the resonant terms. In the following we calculate the two coefficients a, b. At first the standard form of the flux in the two-dimensional center manifold at the Hopf bifurcation point (22) is transformed into a single complex equation:

$$\dot{z} = \lambda z + h(z, \bar{z}) , \tag{A.1}$$

where $\bar{z}$ denotes complex conjugation, with

$$\lambda = i\omega = i\sqrt{k} ,$$

$$h(z, \bar{z}) = \alpha_1 z^2 + \alpha_2 \bar{z}^2 + \alpha_3 z^3 + \alpha_4 z^2 \bar{z} + \alpha_5 z \bar{z}^2 + \alpha_6 \bar{z}^3 + \mathcal{O}(|z|^4) ,$$

where

$$\alpha_j = x_j + i y_j (j = 1, \ldots, 6)$$

with

$$x_1 = (b_y - 2a_x)/4 \quad y_1 = (2a_y - b_x)/4 ,$$

$$x_2 = (-2a_x - b_y)/4 \quad y_2 = (2a_y + b_x)/4 ,$$

$$x_3 = (c_x + d_y - e_x - f_y)/8 \quad y_3 = (f_x + c_y - d_x - e_y)/8 ,$$

$$x_4 = (3c_x + e_x + d_y + 3f_y)/8 \quad y_4 = (3c_y + e_y - d_x - 3f_x)/8 ,$$

$$x_5 = (3c_x + e_x - d_y - 3f_y)/8 \quad y_5 = (d_x + 3f_x + 3c_y + e_y)/8 ,$$

$$x_6 = (c_x + f_y - e_x - d_y)/8 \quad y_6 = (d_x + c_y - f_x - e_y)/8$$

with

$$a_x = 2k^3 k_2/n_1 \quad a_y = 2k^{3/2} k_2 (1 + 2k)/n_1 ,$$

$$b_x = 2k^{5/2} k_2 (1 - k)/n_1 \quad b_y = 2k k_2 (-1 - k + 2k^2)/n_1 ,$$

$$c_x = 8k^5 k_2^2 (2 + k)/n_2 \quad c_y = -8k^{7/2} k_2^2 (2 + 5k + 2k^2)/n_2 ,$$

$$d_x = 4k^{9/2} k_2^2 (-4 + 7k + 6k^2 + k^3)/n_2$$
$$d_y = 4k^3 k_2^2 (4 + k - 20k^2 - 13k^3 - 2k^4)/n_2 ,$$

$$e_x = 4k^4 k_2^2 (1 - 8k - 3k^2)/n_2$$
$$e_y = 4k^{5/2} k_2^2 (-1 + 6k + 19k^2 + 6k^3)/n_2 ,$$

$$f_x = 8k^{9/2} k_2^2/n_2 \quad f_y = -8k^3 k_2^2 (1 + 2k)/n_2$$

with

$$n_1 = (1+k)(1+3k+k^2), n_2 = n_1^2(1+6k+k^2).$$

It follows from calculations based on the normal form theorem (see [4]) that the normal form of the flux in the center manifold at a Hopf bifurcation point has the structure given in eq. (24) which in the complex form reads

$$\dot{w} = \lambda w + c_1 w^2 \bar{w} + c_2 w^3 \bar{w}^2 + \ldots \tag{A.2}$$

with $c_j = a_j + ib_j$. We now use the nonlinear coordinate transformation

$$z = w + \psi(w, \bar{w}), \tag{A.3}$$

where ψ is expressed as a Taylor series

$$\begin{aligned}\psi(w,\bar{w}) = {} & \psi_{ww}w^2/2 + \psi_{w\bar{w}}w\bar{w} + \psi_{\bar{w}\bar{w}}\bar{w}^2/2 \\ & + \psi_{www}w^3/6 + \psi_{ww\bar{w}}w^2\bar{w}/2 + \psi_{w\bar{w}\bar{w}}w\bar{w}^2/2 \\ & + \psi_{\bar{w}\bar{w}\bar{w}}\bar{w}^3/6 + \mathcal{O}(|w|^4),\end{aligned} \tag{A.4}$$

to transform eq. (A.1) into the form of (A.2) and so to determine the coefficients c_j by equating the coefficients of the different $|w|^2$–, $|w|^3$–, etc., terms. From (A.3) and (A.2) it follows that

$$\begin{aligned}\dot{z} &= \dot{w} + \psi_w \dot{w} + \psi_{\bar{w}}\dot{\bar{w}} \\ &= \lambda w + c_1 w^2\bar{w} + c_2 w^3 \bar{w}^2 + \ldots \\ &\quad + (\psi_{ww}w + \psi_{w\bar{w}}\bar{w} + 1/2\psi_{www}w^2 + \psi_{ww\bar{w}}w\bar{w} + 1/2\psi_{w\bar{w}\bar{w}}\bar{w}^2 + \mathcal{O}(|w|^3)) \\ &\quad (\lambda w + c_1 w^2\bar{w} + c_2 w^3\bar{w}^2 + \ldots) \\ &\quad + (\psi_{w\bar{w}}w + \psi_{\bar{w}\bar{w}}\bar{w} + 1/2\psi_{ww\bar{w}}w^2 + \psi_{w\bar{w}\bar{w}}w\bar{w} + 1/2\psi_{\bar{w}\bar{w}\bar{w}}\bar{w}^2 + \mathcal{O}(|w|^3)) \\ &\quad (\overline{\lambda w} + \overline{c_1 w^2 \bar{w}} + \overline{c_2 w^3 \bar{w}^2} + \ldots).\end{aligned} \tag{A.5}$$

From (A.1) and (A.3) it follows that

$$\begin{aligned}\dot{z} = {} & \lambda(w+\psi) + \alpha_1(w+\psi)^2 + \alpha_2(\bar{w}+\bar{\psi})^2 + \alpha_3(w+\psi)^3 \\ & + \alpha_4(w+\psi)^2(\bar{w}+\bar{\psi}) + \alpha_5(w+\psi)(\bar{w}+\bar{\psi})^2 + \alpha_6(\bar{w}+\bar{\psi})^3\end{aligned} \tag{A.6}$$

with ψ from (A.4). Since eq. (A.5) must equal (A.6) for the $\mathcal{O}(|w|^2)$-terms follows the equation

$$\begin{aligned}& \lambda\psi_{ww}w^2 + \lambda\psi_{w\bar{w}}w\bar{w} + \psi_{w\bar{w}}\overline{\lambda w}w + \psi_{\bar{w}\bar{w}}\overline{\lambda w}\bar{w} \\ & \quad = \lambda(\psi_{ww}w^2/2 + \psi_{w\bar{w}}w\bar{w} + \psi_{\bar{w}\bar{w}}\bar{w}^2/2) + \alpha_1 w^2 + \alpha_2\bar{w}^2.\end{aligned} \tag{A.7}$$

Equating the coefficients of w^2, $w\bar{w}$ and $\bar{w}^2$ yields

$$\psi_{ww} = 2\alpha_1/\lambda, \tag{A.8}$$

$$\psi_{w\bar{w}} = 0, \tag{A.9}$$

$$\psi_{\bar{w}\bar{w}} = -2\alpha_2/(3\lambda)\,. \tag{A.10}$$

By equating the coefficients of the $w^2\bar{w}$-terms of (A.5), (A.6) it follows that

$$c_1 = 2\alpha_1\psi_{w\bar{w}} + \alpha_2\overline{\psi_{\bar{w}\bar{w}}} + \alpha_4\,, \tag{A.11}$$

and therefore with (A.9), (A.10) the first coefficient of the normal form (A.2):

$$c_1 = -\frac{2i}{3\sqrt{k}}|\alpha_2|^2 + \alpha_4\,. \tag{A.12}$$

Inserting α_2 and α_4 of (A.1) gives the expressions of (28), (29) for $a = \mathrm{Re}(c_1), b = \mathrm{Im}(c_1)$.

Appendix B

In this appendix we show a way of finding functions $a = f_1(k_i)$, $b = f_2(k_i)$, $\ldots, f = f_6(k_i)$ so that $V_s = f(a,b,c,d,e,f)$ of (41) is a Lyapunov function for proving for the parameter region $0 < k < k_3 + k_5$ the global stability in the surroundings of the nontrivial steady state. The aim is reached if a–f are specified in such a way that the function $\dot{V}_s = f(x,y,z)$ of (42) has a local maximum at $\dot{V}_s(\bar{x},\bar{y},\bar{z}) = 0$. As it can be seen from eq. (42) the necessary condition for an extremum

$$\left.\frac{\partial \dot{V}_s}{\partial x}\right|_{(\bar{x},\bar{y},\bar{z})} = 0, \quad \left.\frac{\partial \dot{V}_s}{\partial y}\right|_{(\bar{x},\bar{y},\bar{z})} = 0, \quad \left.\frac{\partial \dot{V}_s}{\partial z}\right|_{(\bar{x},\bar{y},\bar{z})} = 0 \tag{B.1}$$

is always (i.e. for all values of the parameters) fulfilled. The sufficient condition for a local maximum of $\dot{V}_s$ at the steady state reads: All eigenvalues of the matrix

$$F(\bar{x},\bar{y},\bar{z}) = \begin{pmatrix} \dot{V}_{s_{xx}} & \dot{V}_{s_{xy}} & \dot{V}_{s_{xz}} \\ \dot{V}_{s_{xy}} & \dot{V}_{s_{yy}} & \dot{V}_{s_{yz}} \\ \dot{V}_{s_{xz}} & \dot{V}_{s_{yz}} & \dot{V}_{s_{zz}} \end{pmatrix}, \tag{B.2}$$

where $\dot{V}_{s_{xx}}$ denotes $\partial^2\dot{V}_s/\partial x^2$ $(\bar{x},\bar{y},\bar{z})$, etc., of the corresponding quadratic form must be negative. From eq. (42) the matrix elements follow:

$$\dot{V}_{s_{xx}} = 2ek_4\,,$$

$$\dot{V}_{s_{xy}} = -k_3(2ak/k_4 + d) + fk_4\,,$$

$$\dot{V}_{s_{xz}} = k_5(d-e) + 2ck_4\,,$$

$$\dot{V}_{s_{yy}} = -2k_3(dk/k_4 + 2b)\,,$$

$$\dot{V}_{s_{yz}} = 2bk_5 - ek_3k/k_4 - f(k_3 + k_5)\,,$$

$$\dot{V}_{s_{zz}} = 2k_5(f - 2c)\,.$$

Since $F(\bar{x}, \bar{y}, \bar{z})$ is a real symmetric matrix, it has only real eigenvalues. With the help of the lemma of Gerschgorin one can find a number of conditions which are sufficient for only negative eigenvalues of the matrix F.

At first the three elements of the main diagonal must be negative, therefore,

$$e < 0\,, \tag{B.3}$$

$$kd/k_4 + 2b > 0\,, \tag{B.4}$$

$$2c > f\,. \tag{B.5}$$

A calculation shows that it is not possible to find specifications for a–f so that all non-main-diagonal-elements vanish for all parameter values. So one has to look for functions for a–f which guarantee that these elements are small compared with the main diagonal elements:

$$|2ek_4| > |fk_4 - k_3(2ak/k_4 + d)| + |k_5(d - e) + 2ck_4|\,, \tag{B.6}$$

$$\begin{aligned}|2k_3(kd/k_4 + 2b)| >& |fk_4 - k_3(2ak/k_4 + d)| \\ &+ |2bk_5 - ekk_3/k_4 - f(k_3 + k_5)|\,,\end{aligned} \tag{B.7}$$

$$\begin{aligned}|2k_5(f - 2c)| >& |k_5(d - e) + 2ck_4| \\ &+ |2bk_5 - ekk_3/k_4 - f(k_3 + k_5)|\,.\end{aligned} \tag{B.8}$$

If conditions (B.3) to (B.8) are fulfilled, it follows from the lemma of Gerschgorin that all eigenvalues of the form matrix are negative. In order to simplify the calculations we choose

$$e = \frac{2ck_4 + dk_5}{k_5}\,, \tag{B.9}$$

$$a = \frac{(fk_4 - dk_3)k_4}{2k_3k}\,. \tag{B.10}$$

Condition (B.3) now reads

$$2ck_4 < -dk_5\,. \tag{B.11}$$

Conditions (B.4) and (B.5) remain uneffected, (B.6) vanishes and (B.7), (B.8) simplify to

$$|2k_3(kd/k_4 + 2b)| > |2bk_5 - ekk_3/k_4 - f(k_3 + k_5)|\,, \tag{B.12}$$

$$|2k_5(f-2c)| > |2bk_5 - ekk_3/k_4 - f(k_3+k_5)|\,. \tag{B.13}$$

The sufficient condition for the existence of only negative eigenvalues of $F(\bar{x}, \bar{y}, \bar{z})$ now consists of conditions (B.4), (B.5) and (B.9) to (B.13). If now functions of the parameters k_i for b, c, d, f can be formulated which fulfill these conditions and the conditions for the Lyapunov function (38)–(40), then a Lyapunov function $V_s = f(a, b, c, d, e, f)$ that proves the global stability in a finite surroundings of the steady state is found. From (B.9), (B.11) it follows immediately that

$$d = -2ck_4/k_5 - \epsilon_1 \quad (\epsilon_1 > 0) \tag{B.14}$$

and therefore

$$e = -\epsilon_1\,. \tag{B.15}$$

From (B.4) it follows that $d = -2bk_4/k + \epsilon_2$, $(\epsilon_2 > 0)$, and therefore with (B.14)

$$b = k(c/k_5 + (\epsilon_1 + \epsilon_2)/(2k_4)) > 0\,. \tag{B.16}$$

Condition (B.5) is fulfilled with

$$f = 2c - \epsilon_3 \quad (\epsilon_3 > 0) \tag{B.17}$$

and so (B.10) reads

$$a = k_4/(2k_3k)(k_4(2c-\epsilon_3) + k_3(2ck_4/k_5 + \epsilon_1))\,. \tag{B.18}$$

With (B.14) to (B.18) conditions (B.12), (B.13) read

$$2kk_3\epsilon_2 > |(k_3+k_5)(\epsilon_1 k + \epsilon_3 k_4 - 2ck_4) + k(\epsilon_2 k_5 + 2ck_4)|\,, \tag{B.19}$$

$$2k_4k_5\epsilon_3 > |(k_3+k_5)(\epsilon_1 k + \epsilon_3 k_4 - 2ck_4) + k(\epsilon_2 k_5 + 2ck_4)|\,. \tag{B.20}$$

If one chooses

$$c = \frac{\epsilon_2 kk_5 + \epsilon_3 k_4(k_3+k_5)}{2k_4(k_3+k_5-k)}\,, \tag{B.21}$$

one has fulfilled the necessary conditions $c > 0$ and (because of $2c > \epsilon_3$) $a > 0$ [cf. (38)] and (B.19), (B.20) simplify to

$$2k_3\epsilon_2 > (k_3+k_5)\epsilon_1\,, \tag{B.22}$$

$$2k_4k_5\epsilon_3 > k(k_3+k_5)\epsilon_1\,. \tag{B.23}$$

It follows immediately that

$$\epsilon_2 = (k_3+k_5)\epsilon_1/(2k_3) + \tilde{\epsilon}_2 \quad (\tilde{\epsilon}_2 > 0)\,, \tag{B.24}$$

$$\epsilon_3 = k(k_3+k_5)\epsilon_1/(2k_4k_5) + \tilde{\epsilon}_3 \quad (\tilde{\epsilon}_3 > 0)\,. \tag{B.25}$$

In order to simplify the calculations we choose

$$\epsilon_1 = 1/(k_3 + k_5), \quad \tilde{\epsilon}_2 = 1/(2k_3), \quad \tilde{\epsilon}_3 = k/(2k_4k_5), \tag{B.26}$$

which gives the results (43)–(48). A further calculation yields that the necessary conditions for a Lyapunov function (39), (40) are always fulfilled. Therefore, the global stability in the surroundings of the nontrivial steady state in the parameter range $0 < k < k_3 + k_5$ is proven.

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